COUPLING NOISE EFFECT IN SELF–SYNCHRONIZING WIRELESS SENSOR NETWORKS

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ABSTRACT

Recent results have shown that the mathematical tools considered for modelling populations of coupled oscillators appearing in nature provide an appealing framework for designing self-syncronizing sensor networks. Trendy signal processing applications take advantage of these works by coupling the sensors in order to design reliable decision/estimation networks based on cheap and unreliable sensors. In this work, we extend those results to take into account that the coupling function might suffer from noise due to the need of estimating the states of the nearby sensors.

The novelty of this paper is the introduction of the concept of *frustration* in the design of wireless sensor networks. Frustration implies that synchronization is only possible up to a certain variance standstill floor. We provide the analytic expression of this floor and discuss some limiting cases. In order to assess the performance of the self–synchronizing network, we propose a simple signal model for the transmission of states from node to node and study its Cramér–Rao Bound and the asymptotically efficient Maximum Likelihood estimator. Taking into consideration these achieved estimation variances, computer simulation results are provided discussing the coupling noise effect and the obtained theoretical lower bound.

1. INTRODUCTION

Synchronization appears spontaneously in many biological systems. In Malaysia, thousands of fireflies flash in unison each night, being one of the most wonderful phenomenons of nature. Although they initially flash incoherently, synchronization somehow emerges without the need of a master clock or a leading firefly. Many efforts have been devoted to the study of this and other biological systems such as the chorusing crickets and the firing of neurons. Another system highly reliable and stable (hopefully), composed of many unreliable coupled oscillators, is the one driven by the cardiac pacemaker cells. Each pacemaker cell has an unstable period. However, the human heart is one of the most stable master clocks of nature. But, how does synchronism emerge from disorder? This question has triggered research, attempting at modeling the coupling mechanisms that allow a population of oscillators to synchronize spontaneously. Peskin proposed a pulse–coupled model for pacemakers cells [1], in which all oscillators are assumed to be identical. Later on, it was proved that Peskin's model inevitably synchronizes [2]. For non–identical oscillators, Winfree [3] studied the case of phase–coupled oscillators. He showed that when the coupling between oscillators is small w.r.t. the spread of natural frequencies of each oscillator, the system does not synchronize. The system remains in an incoherent state until the coupling is increased up to a certain threshold, in which clusters of oscillators appear to be synchronized. For high coupling strengths, the whole network of oscillators becomes frequency–locked. Kuramoto proposed a solvable and analyzable nonlinear model for phase–coupled oscillators [4].

The first attempt to merge the modelling of biological oscillators and the design of wireless sensor networks (WSN) is due to Hong and Scaglione [5]. They considered Peskin's model to design sensors as pulse–coupled oscillators. Its main drawback is that information is encoded as a temporal shift, which can be jeopardized by propagation delays causing an unsolvable ambiguity. Recently, Barbarossa proposed the use of Kuramoto's model to design WSN as a population of phase–coupled oscillators [6, 7], showing appealing synchronization properties. In this work, we extend the [6, 7] to take into account that mutual coupling suffers from inherent errors due to the need of estimating the phases of nearby sensors.

This paper is organized as follows. In Section 2, we expose the model that considers this coupling noise effect and introduce the concept of *frustration*. Frustration implies that synchronization is only possible up to a certain variance standstill floor. The analytical expression of the minimum achievable variance is obtained and limiting cases are studied. This constitutes a novel result in the design of self–synchronizing WSN in the sense that this work quantifies the error committed due to the estimation of the phases of nearby sensors for a general case. In Section 3, we propose a simple signal scenario to transmit phases between nearby sensors. We consider the Cramér-Rao Bound (CRB) of that parameter and the Maximum Likelihood Estimator (MLE) for this scenario. The behavior of the self–synchronizing network is

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studied considering the effects of incoherent mutual couplings between sensors and it is compared to the theoretical bound.

2. SELF–SYNCHRONIZING COUPLED OSCILLATORS AND FRUSTRATION

We consider a WSN composed of N nodes, spatially distributed according to an arbitrary topology. Each node senses a physical parameter, maps the estimate into a variable ω_i and initializes a dynamical system. We consider an extension of Kuramoto's model [4] to describe the time evolution of the states of each sensor. The dynamical system of the i–th sensor is coupled with the state of other sensors as

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{c_i}\sum_{j=1}^{N} a_{ij} f\left(\theta_j(t) - \theta_i(t)\right), i = 1, \ldots, N \quad (1)$$

where $\theta_i(t)$ represents the state of the i–th sensor, randomly initialized within the interval $(-\pi, \pi]$ and the function $f(\cdot)$ is monotonically increasing, nonlinear and odd. c_i quantifies the degree of confidence of the i–th sensor in its own estimate, thus determining the degree of adaptation to other sensors' states. K is a control loop gain, which drives the network of oscillators to global synchronization, partial synchronization or no synchronization at all [6, 8]. The coefficient $a_{ij} \geq 0$ takes into account the coupling strength between each pair of sensors. Notice that $a_{ij} = 0$ iff the pair of sensors has null connectivity or $i = j$. In the sequel we assume that the links between two nodes are symmetric, i.e. $a_{ij} = a_{ji}$.

Notice that the dynamical system presented in equation (1) is coupled to the states of nearby sensors, $\theta_j(t)$. However, in a practical scenario these states are not perfectly known but estimated. This fact must be considered in the coupling model, thus extending equation (1) to

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{c_i}\sum_{j=1}^{N} a_{ij} f(\hat{\theta}_{j|i}(t) - \theta_i(t)), i = 1, \ldots, N \quad (2)$$

being $\hat{\theta}_{j|i}(t)$ the estimation performed by the i–th sensor of the state of the j–th sensor. This fact introduces a new ingredient in the synchronization behavior: *frustration* [8]. The classical concept of frustration stands for constant biases in the coupling function. Here we consider stochastic frustration due to the randomness introduced by the estimates $\hat{\theta}_{j|i}(t)$. Frustration implies that the states coupling differences are incoherent, i.e. $\hat{\theta}_{j|i}(t) - \theta_i(t) \neq 0$ even if two sensors are synchronized having identical states. If we consider the MLE of the state of sensor j, for large sample sizes it is asymptotically distributed as

$$\hat{\theta}_{j|i}(t) \sim \mathcal{N}(\theta_j(t), \mathrm{CRB}_i(\theta_j(t))) \quad (3)$$

where $\mathrm{CRB}_i(\theta_j(t))$ is the Cramér-Rao Bound (CRB) on variances, which is a lower bound for any unbiased estimator. Hence, if σ_{ij}^2 denotes the variance of the estimate of sensor j's phase by the i-th sensor, it follows that $\sigma_{ij}^2 \geq \mathrm{CRB}_i(\theta_j(t))$.

We define synchronization as the network state where all sensors oscillate with the same pulsation, i.e. $\dot{\theta}_i(t) = \dot{\theta}^*(t), \forall i$. In the abscence of coupling noise, a well known result [6, 7] is the common oscillating pulsation. Considering the oddness of $f(\cdot)$, multiplying both sides of (1) by c_i and taking the summation over i, we obtain that

$$\dot{\theta}^*(t) \doteq \left(\sum_{i=1}^{N} c_i\omega_i\right) \cdot \left(\sum_{i=1}^{N} c_i\right)^{-1} \quad (4)$$

However, when we introduce noise due to the estimation of the phases of other sensors this result is not correct anymore. In this case, operating as in equation (4), the common pulsation when we consider frustration results in

$$\begin{aligned}
\dot{\theta}_f^*(t) &= \frac{\sum_{i=1}^{N} c_i\omega_i}{\sum_{i=1}^{N} c_i} + \frac{K}{\sum_{i=1}^{N} c_i}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} f(\hat{\theta}_{j|i}(t) - \theta_i(t)) \\
&= \dot{\theta}^*(t) + \frac{K}{\sum_{i=1}^{N} c_i}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} f(\Psi_{ij}(t) + \nu_{ij}(t)) \\
\Psi_{ij}(t) &= \theta_j(t) - \theta_i(t) \quad (5)
\end{aligned}$$

where we have defined the phase difference as $\Psi_{ij}(t)$ and $\nu_{ij}(t)$ is the error committed by sensor i when estimating the j–th state. $\nu_{ij}(t)$ is a zero-mean Gaussian r.v., considered i.i.d. for each ij–pair, whose variance depends upon the achieved estimation performance, which we consider equal for all sensors, i.e. $\sigma^2 \doteq \sigma_{ij}^2, \forall i, j$. Notice that in the case of no coupling noise ($\sigma^2 = 0$), equation (5) equals (4). In what follows we choose $f(\cdot) = \sin(\cdot)$ as the coupling function, which reduces to Kuramoto's model when $a_{ij} = 1, \forall i, j$. In addition, K is choosen such that the network achieves a global synchronization state [6].

We are interested in quantifying the variance of the error between the expected synchronized pulsation and the actual, i.e. $|\dot{\theta}_f^*(t) - \dot{\theta}^*(t)|^2$. For the sake of clarity, we start our discussion with the case of two coupled oscillators and then extend the results for the case of arbitrary N.

2.1. Two coupled oscillators case

When we consider two oscillators ($N = 2$), the common oscillation frequency is provided by

$$\begin{aligned}
\dot{\theta}_f^*(t) &= \dot{\theta}^*(t) + \xi(t) \quad (6) \\
\xi(t) &= \frac{Ka}{\sum_{i=1}^{N} c_i}\left(\sin(\Psi(t) + \nu_1(t)) + \sin(-\Psi(t) + \nu_2(t))\right)
\end{aligned}$$

where we have defined $\Psi(t) \doteq \Psi_{12}(t) = -\Psi_{21}(t)$, $\nu_1 \doteq \nu_{12}$, $\nu_2 \doteq \nu_{21}$ and $a \doteq a_{12} = a_{21}$ for the sake of clarity. $\xi(t)$ represents the noise contribution to the coupling pulsation due to the coupling noise effect. In order to evaluate the degradation of $\dot{\theta}^*_f(t)$ w.r.t to the ideal value $\dot{\theta}^*(t)$, we study the variance of $\xi(t)$, which will provide a synchronization misadjustment bound.

$$\begin{aligned} \sigma^2_\xi &= E\left\{\xi^2(t)\right\} = K^2 a^2 \left(\sum_{i=1}^{N} c_i\right)^{-2} E\left\{P(t)\right\} \quad (7) \\ P(t) &= \left(\sin(\Psi(t)+\nu_1(t)) + \sin(-\Psi(t)+\nu_2(t))\right)^2 \end{aligned}$$

after some straightforward mathematical manipulations we obtain that

$$\begin{aligned} P(t) &= \left(\sin(\Psi(t))\left(\cos(\nu_1(t)) - \cos(\nu_2(t))\right)\right. \\ &+ \left.\cos(\Psi(t))\left(\sin(\nu_1(t)) + \sin(\nu_2(t))\right)\right)^2 \\ &= \sin^2(\Psi(t))\left(\cos(\nu_1(t)) - \cos(\nu_2(t))\right)^2 \\ &+ \cos^2(\Psi(t))\left(\sin(\nu_1(t)) + \sin(\nu_2(t))\right)^2 \\ &+ \sin(\Psi(t))\cos(\Psi(t))\left(\cos(\nu_1(t)) - \cos(\nu_2(t))\right) \\ &\cdot \left(\sin(\nu_1(t)) + \sin(\nu_2(t))\right) \quad (8) \end{aligned}$$

When taking the expectation of $P(t)$ we use that $E\left\{\cos(\alpha)\sin(\beta)\right\} = 0$ if α and β are independent Gaussian r.v.'s and β has zero mean. Thus, after plain but lengthy calculations we obtain that

$$\begin{aligned} \overline{P}(\Psi(t),\sigma^2) &= E\Big\{1 + \frac{1}{2}\left(\cos(2\nu_1(t)) + \cos(2\nu_2(t))\right) \\ &\cdot \left(\sin^2(\Psi(t)) - \cos^2(\Psi(t))\right) \\ &- 2\sin^2(\Psi(t))\cos(\nu_2(t))\cos(\nu_1(t))\Big\} \end{aligned}$$

where $\overline{P}(\Psi(t),\sigma^2) \doteq E\left\{P(t)\right\}$. Being $\nu(t) \sim \mathcal{N}(0,\sigma^2)$, we define $g(\sigma^2) = E\left\{\cos(\nu(t))\right\}$. We can express the synchronization error variance in equation (7) as a function of σ^2, i.e. the variance of the estimator of phases.

$$\begin{aligned} \sigma^2_\xi &= K^2 a^2 \left(\sum_{i=1}^{N} c_i\right)^{-2} \overline{P}(\Psi(t),\sigma^2) \quad (9) \\ \overline{P}(\Psi(t),\sigma^2) &= 1 + 2\sin^2(\Psi(t))(g(4\sigma^2) - g^2(\sigma^2)) - g(4\sigma^2) \end{aligned}$$

The latter provides a closed form equation of the variance of $\xi(t)$, since it follows from [9, eq. 45.3.20.70] that

$$g(\sigma^2) = e^{-\frac{\sigma^2}{2}}. \quad (10)$$

A study of the limiting behavior of equation (9) determines the bounds of expected variance of the synchronization error. Notice that $g(4\sigma^2) - g^2(\sigma^2) < 0, \forall\sigma^2$ and that $E\left\{P(t)\right\} > 0$ by definition. Thus, the maximum value of $\overline{P}(\Psi(t),\sigma^2)$ is achieved when $g(4\sigma^2) - g^2(\sigma^2)$ does not contribute, i.e. when $\Psi(t) = m\pi, m \in \mathbb{Z}$, we obtain

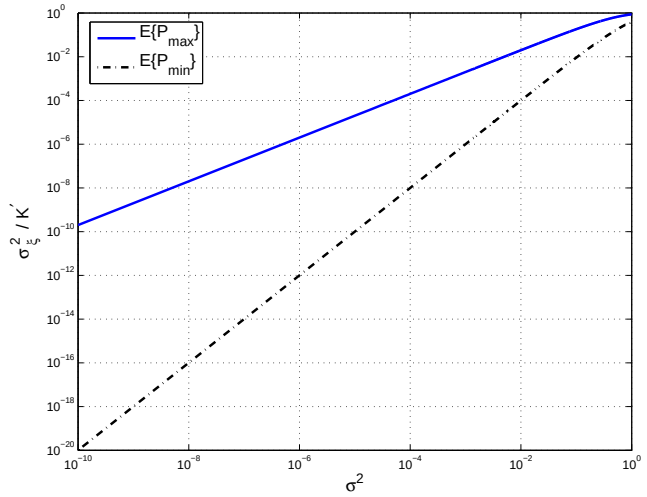

Fig. 1. Variance bounds of $E\left\{|\dot{\theta}^*_f(t) - \dot{\theta}^*(t)|^2\right\}$ as a function of σ^2.

$\overline{P}_{\max} = (1 - e^{-2\sigma^2})$. On the other hand, when $\Psi(t) = (2m+1)\pi/2, m \in \mathbb{Z}$, we obtain the minimum achievable variance value as $\overline{P}_{\min} = (1 - e^{-\sigma^2})^2$. Hence, the minimum achievable variance of the common pulsation of the network lies between the following limits

$$\begin{aligned} K' a^2 (1 - e^{-\sigma^2})^2 \le \sigma^2_\xi \le K' a^2 (1 - e^{-2\sigma^2}) \\ E\left\{|\dot{\theta}^*_f(t) - \dot{\theta}^*(t)|^2\right\} \ge \sigma^2_\xi \quad (11) \end{aligned}$$

where we have defined $K' = K^2 \left(\sum_{i=1}^{N} c_i\right)^{-2}$.

It is a revealing result that, when there is a noise error in the mutual coupling function, the worst case occurs when the two sensors appear to be phase–synchronized or with opposite phases, i.e. $\Psi(t) = \{0, \pi\}$ respectively. In contrast, the lower variance is achieved when the sensors are in quadrature, $\Psi(t) = \pm\pi/2$.

Notice that σ^2_ξ depends on the actual realization of the dynamical system. Thus, the variance bounds are useful to determine the lower and upper values of the MSE between the expected common synchronization pulsation ($\theta^*(t)$) and the achieved pulsation, represented in Figure 1. However, from equations (9) and (10) an instantaneous variance bound can be computed for each specific realization.

2.2. N coupled oscillators case

In this section, we extend the results of σ^2_ξ bounds to the general case of a population of N oscillators. In this case, we have that

$$\xi(t) = \sqrt{K'} \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\sin(\Psi_{ij}(t) + \nu_{ij}(t)), \quad (12)$$

and

$$\sigma_\xi^2 = K' \cdot E\left\{\left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\sin(\Psi_{ij}(t)+\nu_{ij}(t))\right)^2\right\}$$
$$= K' \cdot \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} a_{ij}^2 \cdot E\left\{\overline{P}(\Psi_{ij}(t),\sigma^2)\right\} \quad (13)$$

provides the instantaneous variance value for a given realization as a function of each $\Psi_{ij}(t)$, a_{ij} and σ^2.

We consider networks of oscillators which, altough totally connected, need not be fully meshed. As previously defined, $a_{ij} = 0$ means oscillators i and j are not connected and $a_{ii} \doteq 0$, $\forall i$. In addition, we can assume that $a \doteq a_{ij}, \forall a_{ij} \neq 0$. Taking this into account, there are terms in equation (13) that must be neglected when aiming at studying limiting cases because it may happen that some ij–pairs do not contribute to σ_ξ^2. As a practical choice, we consider that each oscillator is coupled with its d nearest neighbors in subsequent numerical simulations, $1 \leq d \leq N-1$. Defining N_p as the total number of pairs for which $a_{ij} \neq 0$, the bounds on the common pulsation variance in the N–oscillators case amount to

$$K'a^2N_p(1-e^{-\sigma^2})^2 \leq \sigma_\xi^2 \leq K'a^2N_p(1-e^{-2\sigma^2}). \quad (14)$$

For a fully–meshed network, $N_p = N(N-1)/2$.

3. DEGRADATION OF SYNCHRONISM DUE TO THE COUPLING NOISE EFFECT

As mentioned in Section 2, the self–synchronization performance of the coupled system is degraded because of the need of estimating states from other sensors. Since the CRB provides the minimum expected variance due to the estimation of those states ($\sigma^2|_{min}$), we can evaluate the impact of coupling noise in the network for a given estimator.

We now consider a specific setting in order to obtain values of expected estimation variance σ^2. We consider that sensor j transmits its state in a sinusoidal signal and that sensor i receives a corrupted version of it:

$$y_{ij}[\ell] = b_{ij}\cos(2\pi f_j \ell + \theta_j) + n[\ell], \ell = 0,\ldots,L-1 \quad (15)$$

where L is the number of samples considered. The unknown parameters to estimate in the received signal are the amplitude ($b_{ij} > 0$), natural oscillating frequency ($\omega_j = 2\pi f_j$ and $0 < f_j < 1/2$) and phase (θ_j). The received signal is corrupted by a zero–mean additive white Gaussian noise with unit variance, so that the signal-to-noise ratio (SNR) can be straightforwardly obtained as $\text{SNR}_{ij} = b_{ij}^2/2$. Without loss of generality, we consider that the multiple access technique is implemented by a higher layer protocol, being the estimates computed independently.

We now consider the CRB of the parameters in (15) and their MLE, which turns to be a simple operation that facilitates its implementation in cheap sensors. These are well-known results which can be computed analogously as done in [10, pg. 56,193], resulting in the following variance bounds

$$var\left(\hat{b}_{ij|i} - b_{ij}\right) \geq \frac{2\sigma^2}{L}$$
$$var\left(\hat{f}_{j|i} - f_j\right) \geq \frac{12}{\text{SNR}_{ij}(2\pi)^2L(L^2-1)}$$
$$var\left(\hat{\theta}_{j|i} - \theta_j\right) \geq \frac{2(2L-1)}{\text{SNR}_{ij}L(L+1)} \quad (16)$$

and ML estimates

$$\hat{f}_{j|i} = \arg\max_f \left\{\frac{2}{L}\left|\sum_{\ell=0}^{L-1} y_{ij}[\ell]e^{-j2\pi f_j \ell}\right|^2\right\} \quad (17)$$
$$\hat{b}_{ij|i} = \frac{2}{L}\left|\sum_{\ell=0}^{L-1} y_{ij}[\ell]e^{-j2\pi \hat{f}_{j|i}\ell}\right|$$
$$\hat{\theta}_{j|i} = \arctan\left\{-\frac{\sum_{\ell=0}^{L-1} y_{ij}[\ell]\sin(2\pi\hat{f}_{j|i}\ell)}{\sum_{\ell=0}^{L-1} y_{ij}[\ell]\cos(2\pi\hat{f}_{j|i}\ell)}\right\}$$

With these results we can evaluate the degradation due to the estimation of other sensors' states in the coupled dynamical system. Notice that these variances depend both on the number of samples and on the SNR of the received signal.

For the following simulations, we consider a network composed of 40 sensors and $d = 4$. We plot the average frequency MSE in the network w.r.t. $\dot{\theta}^*(t)$ given by equation (4), i.e. $E\left\{|\dot{\theta}_f^*(t) - \dot{\theta}^*(t)|^2\right\}$. Its theoretical bound, analytically computed from equation (13), is plotted as dash-dotted curves. The case of *no frustration* is presented as a benchmark, this is the case where $\hat{\theta}_{j|i}(t) = \theta_j(t) \Leftrightarrow \sigma^2 = 0$. In Figure 2 the performance for a number of SNR values is presented, considering the variance provided by the CRB when $L = 1000$ samples. The average frequency error diverges from the optimal curve to a standstill error floor. The level of that floor depends on $\sigma^2 \propto 1/\text{SNR}$. As predicted by the bounds on σ_ξ^2, when σ^2 increases, the minimum achievable variance increases also. For $L = 100$ samples, the performance degradation is higher for the same SNR values, since the variance of the estimates increases, as predicted by the CRB. In any case, the achieved MSE attains the bound computed in Section 2. Thus, the obtained closed-form solution of σ_ξ^2 constitutes a framework for quantifying the degree of frustration in self-synchronizing networks.

4. CONCLUSIONS

Biological mutually phase-coupled oscillators concepts appeared to be a powerful tool for designing WSN. Recent lit-

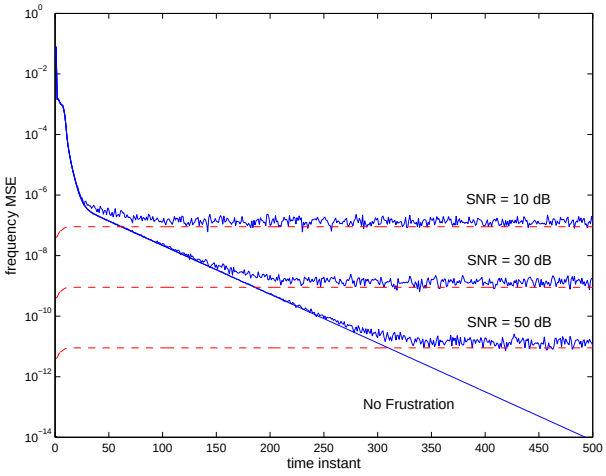

Fig. 2. Average network frequency MSE for the case of no coupling noise (perfect knowledge of states of nearby sensors) for several SNRs ($L = 1000$).

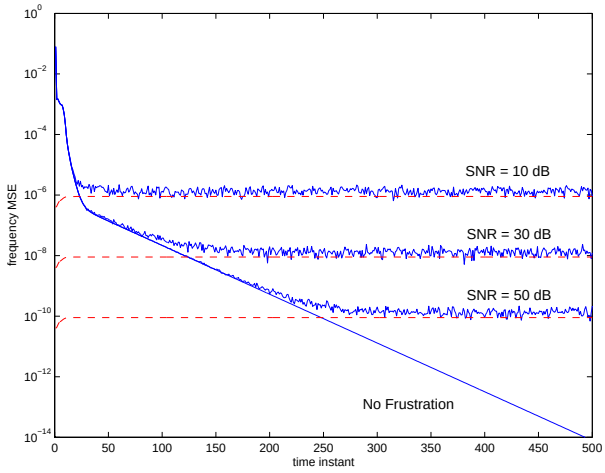

Fig. 3. Average network frequency MSE for the case of no coupling noise (perfect knowledge of states of nearby sensors) for several SNRs ($L = 100$).

erature, mainly due to Barbarossa, has merged these two topics to configure sensor networks with self–organizing capabilities. This paper has presented an extension of the latter by considering that the knowledge of the phases of nearby sensors cannot be assumed perfect in a WSN application. Instead, the phases must be estimated and the variances of those estimates drive the network to a synchronization standstill error floor. We have introduced the concept of *frustration* in the design of self-organizing WSN, which considers that $\hat{\theta}_{j|i}(t) - \theta_i(t) \neq 0$ even if two sensors are synchronized having identical states due to the estimation variance. We have determined the common pulsation error due to frustration as a function of the variance of the phase estimates and bounds of the power of this error have been analytically obtained. Aiming at quantifying the degree of frustration in an implementable system, we have proposed a signal model for the transmission of states of sensors to its neighbors. We have studied the MLE and CRB as a benchmark, showing that the MSE results obtained attain the theoretical bound presented in this paper.

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