

## UNIVERSAL BEHAVIOUR IN A "MODULATED" LOGISTIC MAP

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Received 26 November 1986; revised manuscript received 28 August 1987; accepted for publication 5 October 1987

Communicated by D.D. Holm

We study the period-doubling bifurcations to chaos in a logistic map with a nonlinearly modulated parameter and show that the bifurcation structure is modified significantly. Using the renormalisation method due to Derrida et al. we establish the universal behaviour of the system at the onset of chaos.

Simple deterministic nonlinear systems have attracted a lot of attention during the last few years [1], not only because they are very rich in their dynamical behaviour but they also serve as effective mathematical models to describe several physical systems [2]. Experimental studies on the chaotic behaviour were stimulated by Feigenbaum's discovery [3,4] that the systems show universal behaviour at the onset of chaos. The universal constants predicted by Feigenbaum were measured in several experiments, thus enabling a quantitative analysis of the transition to chaos.

Several examples are known in which the system shows a period-doubling route to chaos, where the details of the transition are represented by a bifurcation structure. Recently, there has been an increased interest in the study of different kinds of laser systems by modulating one of its physical parameters [5] and it has been shown that the output can be periodic as well as chaotic depending on the strength of the modulation [6]. Moreover, the analysis of the bifurcation structure by introducing a time dependence to the parameter has been a subject matter for the last few years and it is known that this time dependence can induce dramatic changes in the bifurcation diagram [7]. Recently, Kapral and Mandel [8] studied the bifurcation structure of a non-autonomous quadratic map in which the control parameter is assumed to vary linearly in time, and showed that this time dependence delayed the onset of bifurcations in the system.

In this Letter, we study the period-doubling bifurcations to chaos in the following map:

$$\begin{aligned}x_{t+1} &= 4\lambda_t x_t (1 - x_t), \\ \lambda_{t+1} &= 4\mu \lambda_t (1 - \lambda_t).\end{aligned}\quad (1)$$

That is, we have considered a situation where the value of the parameter  $\lambda$  at any instant is a simple nonlinear function of its value in the previous instant. We now that map (1), which we call a "modulated" logistic map, has a bifurcation structure which is interesting in many ways. The role of the control parameter is now played by  $\mu$ , which can be thought of as the strength of the modulation.

We know that for  $0 < \mu < 0.75$ , there is a single attracting fixed point for  $\lambda_t$  and hence for  $x_t$  also. Therefore, to compute the bifurcation structure, we start from a parameter value  $\mu = 0.7$  and increase it by steps of 0.01, always using an initial condition for  $(x_t, \lambda_t)$  in the interval  $[0, 1]$ , say  $(0.3, 0.3)$ . The important asymptotic values for  $\lambda_t$ ,  $x_t$  and the corresponding parameter  $\mu$  are collected in table 1.

When we plot these values against  $\mu$ , a three-dimensional bifurcation diagram results. But the essential modification in the structure and the new features that appear due to the modulation of the parameter can be clearly shown by taking a two-dimensional projection of the diagram in the  $(x_t, \mu)$  plane, which is shown in fig. 1.

It is clear that even from the first bifurcation onwards the behaviour is quite different from that of

Table 1  
Asymptotic values of  $\lambda_i$  and  $x_i$ .

$\mu$	$\lambda_i$	$x_i$
0.7	0.6428571	0.6111111
0.74	0.6621622	0.6224488
0.75	0.6683250	0.6202936
	0.6649998	0.6296404
0.78	0.5475750	0.7489202
	0.7729381	0.4118614
0.8	0.5130444	0.7539450
	0.7994550	0.3807031
0.84	0.4623756	0.7534620
	0.8352432	0.3435580
		0.7534756
		0.3435452
0.85	0.4519632	0.7340323
	0.8421544	0.3529470
		0.7693097
		0.3208439
0.87	0.3950656	0.8688380
	0.8316804	0.1800840
	0.4871584	0.4912024
	0.8694256	0.4870072
0.89	0.3488245	0.8894232
	0.8086390	0.1372268
	0.5508806	0.3829568
	0.8807835	0.5206940
	0.3738138	0.8792732
	0.8333148	0.1587240
	0.4944891	0.4450920
	0.8898920	0.4885256

the logistic map. One novel aspect of the diagram is the crossing over of the inner bifurcation branches in the 4-cycle region. Even though the branches appear to cut each other, giving the appearance of a 3-cycle in the 4-cycle region, it is not so because we are only considering the projection of the three-dimensional diagram in a two-dimensional plane. The crossing over of the bifurcation branches is the result of a significant change in the asymptotic behaviour of the system from the normal one, in a small range of the parameter, say  $\mu=0.86$  to  $0.87$ , where the branches appear to be very steep.

Another important feature is that the bifurcations occur earlier than in the case of the logistic map from the second bifurcation onwards. Even though the difference becomes less pronounced as we go to the higher and higher bifurcations, it is clearly evident in the case of the 2-cycle, which becomes unstable somewhere around  $\mu=0.845$ . Comparing with the

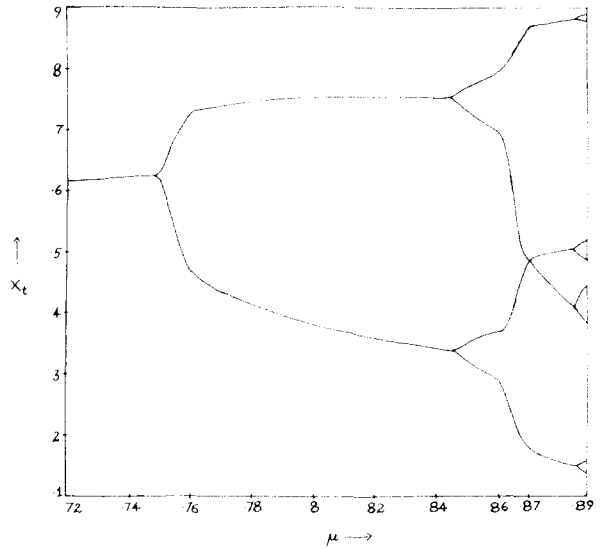


Fig. 1. Bifurcation diagram of map (1). The branches cross over each other in the 4-cycle region resulting in a complete modification of the structure. Note also the asymmetry of the figure.

results obtained by Kapral and Mandel [8], this shows that linear and nonlinear variations of the parameter can have an entirely different effect on the asymptotic behaviour of the system.

Finally, we note a peculiar aspect of our bifurcation diagram. It is easily seen that fig. 1 lacks the symmetry of the bifurcation structure of the logistic map. There is a marked difference in the bifurcations between the upper and lower arms of our bifurcation tree. All the bifurcations of the upper branch appear to be asymmetric whereas that of the lower branch are somewhat symmetric.

All these novel aspects of the diagram must be considered as the effect of nonlinear modulation of the parameter.

We now show that the system has universal behaviour at the onset of chaos. For this, we make use of the renormalisation method due to Derrida et al. [9,10].

Let map (1) be represented as  $T_\mu(x, \lambda)$ . We linearise the map  $T_\mu^{(n)}$  in the neighbourhood of the  $n$ -cycle. The resulting  $2 \times 2$  matrix  $M$  is given by

$$M = \prod_{i=1}^n \begin{pmatrix} 4\lambda_i(1-2x_i) & 4x_i(1-x_i) \\ 0 & 4\mu(1-2\lambda_i) \end{pmatrix}. \tag{2}$$

In the first-order approximation we consider cycles of period 1 and 2.

The stable 1-cycle of map (1) is given by

$$(x^*, \lambda^*) = \left( \frac{3\mu-1}{4\mu-1}, \frac{4\mu-1}{4\mu} \right). \quad (3)$$

Evaluating  $M$  at the fixed point  $(x^*, \lambda^*)$  and taking the eigenvalue equation, we get:

$$\sigma^2 + 2f_1(\mu)\sigma + (8\mu + 2/\mu - 8) = 0, \quad (4)$$

where

$$f_1(\mu) = 2\mu - 1/2\mu. \quad (5)$$

Take  $(x_1^*, \lambda_1^*)$  and  $(x_2^*, \lambda_2^*)$  as the stable 2-cycle of map (1). By completely solving the four equations defining the 2-cycle, we get

$$\lambda_1^* = \frac{4\mu+1}{8\mu} + \frac{[(4\mu+1)(4\mu-3)]^{1/2}}{8\mu}, \quad (6)$$

$$x_1^* = \frac{(5\mu-1)}{2(4\mu-1)} - \frac{1}{2} \left( \frac{(5\mu-1)^2}{(4\mu-1)^2} + \frac{8\mu(4\mu-1)(\mu^2-4\mu-1)}{(4\mu+1)(3\mu-1)\beta} \right)^{1/2}, \quad (7)$$

$$\lambda_2^* = \frac{(4\mu+1)}{8\mu} - \frac{[(4\mu+1)(4\mu-3)]^{1/2}}{8\mu}, \quad (8)$$

$$x_2^* = \frac{(4\mu+1-\mu^2)(4\mu-1)}{(4\mu+1)(3\mu-1)} - \frac{\beta x_1^*}{2(4\mu-1)}, \quad (9)$$

where

$$\beta = (4\mu+1) + [(4\mu+1)(4\mu-3)]^{1/2}. \quad (10)$$

From eqs. (6) and (8) we see that the  $\lambda$ -values are symmetric whereas eqs. (7) and (9) suggest that  $x$ -values are asymmetric resulting in the asymmetry of fig. 1.

As in the case of 1-cycle, evaluating  $M$  at the 2-cycle and taking the eigenvalue equation, we get:

$$f_2(\mu) = \frac{1}{2}(4\mu+1)(4\mu-3) - \frac{4\mu+1}{2\mu^2(4\mu-1)} (A^{1/2} - \mu) - \frac{(4\mu+1)(6\mu-1)\beta}{4\mu^2(4\mu-1)^3} [A^{1/2} - (5\mu-1)] + \frac{(4\mu+1-\mu^2)}{\mu^2(3\mu-1)} (A^{1/2} - 3\mu) - \frac{1}{2}, \quad (11)$$

where

$$A = (5\mu-1)^2 - \frac{8\mu(4\mu-1)^3(4\mu+1-\mu^2)}{(4\mu+1)(3\mu-1)\beta}. \quad (12)$$

It can easily be seen that  $|2f_n(\mu)|$  is equal to  $|\text{Tr } M|$  evaluated at the  $n$ -cycle, so that the  $n$ -cycle is stable when  $|f_n(\mu)| \leq 1$ .

The basic idea of this renormalisation method is to try to associate at each value of  $\mu$ , a value  $\mu'$  such that  $T_\mu$  looks like  $T_{\mu'}^{(2)}$ . An approximate way to do so is to say that the linearisation of  $T_\mu^{(n)}$  around a point of cycle  $n$  is identical to the linearisation of  $T_{\mu'}^{(2n)}$  around a point of cycle  $2n$ . Therefore:

$$f_n(\mu) = f_{2n}(\mu'). \quad (13)$$

This equation gives an approximate value of  $\mu_\infty$  which is the fixed point of renormalisation:

$$f_n(\mu_\infty) = f_{2n}(\mu_\infty). \quad (14)$$

In the first-order approximation, we have

$$f_1(\mu_\infty) = f_2(\mu_\infty). \quad (15)$$

From extensive numerical calculation, we obtain

$$\mu_\infty = 0.890\dots, \quad (16)$$

which is very close to the  $\mu_\infty$  of the logistic map.

The bifurcation ratio  $\delta$  is given by

$$\delta = d\mu/d\mu' |_{\mu_\infty} \quad (17)$$

Interestingly enough, we obtain  $\delta$  as

$$\delta = 4.4339741\dots, \quad (18)$$

which is in close agreement with Feigenbaum's constant, considering the fact that we are taking a first-order approximation. This establishes the universal behaviour of the system at the onset of chaos.

One of the authors (KPH) is grateful to CSIR, New Delhi, for financial assistance.

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