AN ANALOGUE OF THE LOGISTIC MAP IN TWO DIMENSIONS

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This is a sequel to our earlier work on the modulated logistic map. Here, we first show that the map comes under the universality class of Feigenbaum. We then give evidence for the fact that our model can generate strange attractors in the unit square for an uncountable number of parameter values in the range $\mu_\infty < \mu < 1$. Numerical plots of the attractor for several values of μ are given and the self-similar structure is explicitly shown in one case. The fractal and information dimensions of the attractors for many values of μ are shown to be greater than one and the variation in their structure is analysed using the two Lyapunov exponents of the system. Our results suggest that the map can be considered as an analogue of the logistic map in two dimensions and may be useful in describing certain higher dimensional chaotic phenomena.

1. Introduction

Recently, we introduced a "modulated" logistic map

$$X_{t+1} = 4\lambda_t X_t (1 - X_t) ,$$
$$\lambda_{t+1} = 4\mu\lambda_t (1 - \lambda_t) , \qquad (1)$$

and established many interesting properties for this system including the universal metric as well as structural properties of unimodal maps [1,2]. The value of the control parameter μ determines the asymptotic behaviour of the map. Note that the map has the following properties analogous to the logistic map:

(a) It consists of two coupled first order difference equations which map the unit square $\{x, \lambda | 0 \leqslant x \leqslant 1,\ 0 \leqslant \lambda \leqslant 1\}$ in R^2 into itself for $\mu \in [0, 1]$ and is continuous over the interval.

(b) It can be included in a one-parameter family of maps.

(c) For a given value of μ, there is a unique attractor for the map that "owns" almost all initial conditions in the unit square. This is true even in the chaotic regime where there are infinitely many different periodic orbits and an uncountable number of asymptotically aperiodic orbits. Also, the infinite number of periodic windows in the chaotic region, some of which are large and some unimaginably small, are all arranged along the parameter axis exactly in the same order as in the case of the logistic map. In the present Letter, we investigate this map in detail using various tools such as renormalisation, Lyapunov exponent and fractal dimension and show that the system possesses many fascinating features.

2. Universal scaling

As we know, most of the interest in the study of chaotic systems was stimulated by the discovery of universal metric properties in quadratic maps by Feigenbaum [3,4]. We have already shown [1] that our map turns chaotic with the Feigenbaum ratio δ. Here we calculate the rescaling coefficient α of our system using the renormalisation method developed by Helleman [5,6]. Since the map (1) is two-dimensional, there exist two rescaling coefficients for the map. Moreover, the coefficient for the variable λ has to be necessarily α. Our aim is to calculate the second one, say α', for the variable X. The basic principle of the method is to look for the local behaviour about a periodic orbit of the map by expanding it about a periodic point, up to and including second order Taylor terms in the deviations. After proper scaling and counting t modulo 2, the quadratic part of the mapping about the periodic orbit becomes identical to the original equation we started with. Since λ is decoupled from X in the map (1), a small variation in X about a periodic point does not lead to a variation in the λ-cycle.

Let (X_t^*, λ_t^*) be a periodic point of the map (1). Taking a small variation ΔX_t for X_t about the periodic orbit, the variational equation for X_t can be written as

$$\Delta X_{t+1} = 4\lambda_t^*(1-2X_t^*)\Delta X_t - 4\lambda_t^*(\Delta X_t)^2 . \tag{2}$$

Putting $t=2\eta+1$ in (2) and taking a periodic orbit of period 2, denoted by

$$(X_{2\eta}^*, \lambda_{2\eta}^*) \equiv (X_1^*, \lambda_1^*), \quad (X_{2\eta+1}^*, \lambda_{2\eta+1}^*) \equiv (X_2^*, \lambda_2^*) ,$$

we get

$$\Delta X_{2\eta+2} = 4\lambda_2^*(1-2X_2^*)\Delta X_{2\eta+1} - 4\lambda_2^*(\Delta X_{2\eta+1})^2 . \tag{3}$$

Now, putting $t=2\eta$ in (2),

$$\Delta X_{2\eta+1} = 4\lambda_1^*(1-2X_1^*)\Delta X_{2\eta} - 4\lambda_1^*(\Delta X_{2\eta})^2 . \tag{4}$$

Putting (4) into (3) and collecting terms up to quadratic in $\Delta X_{2\eta}$, we obtain

$$\begin{aligned}\Delta X_{2\eta+2} &= 4[4\lambda_1^*\lambda_2^*(1-2X_1^*)(1-2X_2^*)]\Delta X_{2\eta} \\ &\quad - 4[16\lambda_1^{*2}\lambda_2^*(1-2X_1^*)^2 + 4\lambda_1^*\lambda_2^*(1-2X_2^*)](\Delta X_{2\eta})^2 \\ &\equiv 4P_\eta \Delta X_{2\eta} - 4Q_\eta(\Delta X_{2\eta})^2 . \end{aligned} \tag{5}$$

Now, taking $Y_\eta = \alpha'\Delta X_{2\eta}$, where $\alpha' = Q_\eta/P_\eta$ we get the "renormalised" mapping

$$Y_{\eta+1} = 4P_\eta Y_\eta(1-Y_\eta) , \tag{6}$$

which is identical to the original equation. The rescaling coefficient α' is given by

$$\alpha' = \frac{4\lambda_1^*(1-2X_1^*)^2 + (1-2X_2^*)}{\cancel{(1-2X_1^*)(1-2X_2^*)}} . \tag{7}$$

Note that α' is determined by the control parameter μ since X^* and λ^* are given in terms of μ. Using the expressions for the stable two-cycle of the map (1) obtained by us [7] in the renormalisation limit μ_∞, we get the rescaling coefficient as $\alpha' = 2.6545422 \sim \alpha$ as a first order approximation. In other words, both the rescaling coefficients of the map (1) are the same and are equal to α. Thus the map shows the universal scaling behaviour in addition to the other universal properties of the logistic map already reported.

3. Strange attractors

Strange attractors, with their fractal structure [8], have played a major role in our understanding of the properties of chaotic dynamical systems since their first discovery by Lorenz [9]. Here our aim is to show the existence of strange attractors for the map (1) for many values of the parameter (in fact uncountable in number, as we shall see below) and to calculate the two important measures characterising them, namely, the fractal and information dimensions.

In order to decide whether an attractor is periodic or chaotic, we must look at the spectrum of Lyapunov exponents (LE) characterising the attractor. For the map (1), we can define two LEs [10], say σ_1 and σ_2, measuring the exponential divergence along the X and λ directions respectively. It is easy to see that σ_2 is the same as the LE of the logistic map. Now, as we see below, σ_1 is always negative independent of the value of μ. This indicates that when $\sigma_2 > 0$, the attractor can possibly be strange due to the stretching and folding of the trajectories on the unit square. We now choose one such value, say, $\mu = 0.895$ and show the corresponding attractor in fig. 1a by plotting 8000 points after the initial 5000 points were discarded. Taking a small region of the attractor indicated by a box in the figure and enlarging it we get fig. 1b and repeating this process once again we get fig. 1c. From the figures, it is clear that the attractor has a self-similar structure, which is in fact very much similar to that of the Hénon attractor [11]. We can also give a simple reason as a support to this numerical evidence. From the Cantor set [8] structure of the λ-values in the interval [0, 1] for $\sigma_2 > 0$, it directly follows that any two arbitrarily close points on the attractor will have another point in between (for a sufficiently large number of iterations) making the attractor self-similar. But it is found that the self-similar structure becomes less pronounced within the limits of numerical precision for larger values of the parameter. The reason for this can be attributed to the observed variation in the values of σ_1 and σ_2, shown in fig. 3 below.

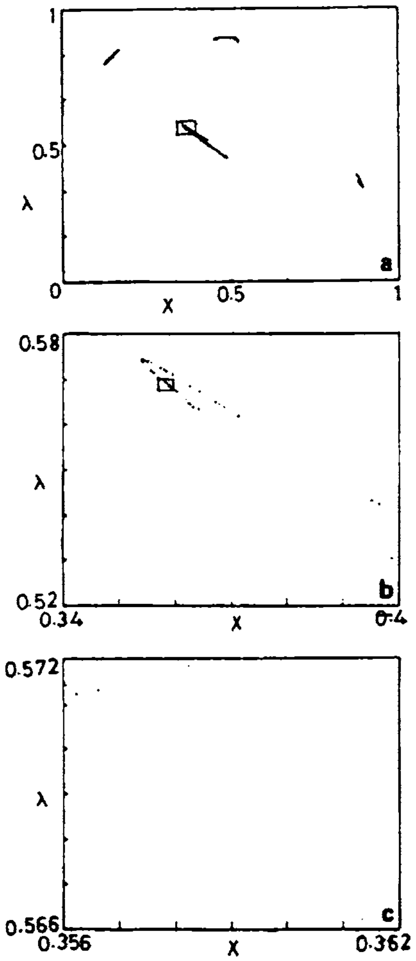


Fig. 1. A plot of the attractor of map (1) for $\mu=0.895$. (b) and (c) were obtained by enlarging the squared regions of the previous figures and clearly display the self-similar structure of the attractor.

We now present a numerical plot of the attractor in fig. 2 for six different values of μ. For μ just beyond μ_∞, the attractor consists of several disjoint sets and finally becomes a single piece for sufficiently large value of μ. This structure variation necessarily reflects the band merging in the logistic map. Now, the values of μ at which $\sigma_2>0$ are uncountable in number and form a set of positive measure on the parameter axis [12,13]. So, in principle, the map can be said to generate an uncountable number of strange attractors in the unit square apart from the infinite number of periodic cycles already shown. Here, we want to make one point clear. We see from fig. 3 below that for sufficiently large values of μ the sum of the LEs is larger than zero implying that the area elements grow on the average. But we call the resulting fractal set a "strange attractor" because it satisfies other properties of an attractor. For example, it is a bounded region to which almost all other initial conditions in the unit square get attracted asymptotically. Moreover, it forms an invariant set every part of which is eventually visited by the iterates.

One important question regarding a strange attractor is its dimension. Even though there are a variety of different definitions of dimension, the most relevant ones are of two types. One depending only on metric properties and the other depending on metric as well as probabilistic properties. The former one is called the Hausdorff or fractal dimension [8] which we denote by D_0 while the latter is the dimension of the natural measure more commonly known as the information dimension and denoted by D_1. Even though one can define an infinite number of generalised dimensions D_q [14] for a strange attractor, it is sufficient to know the above two dimensions for our purpose here. For further details regarding the dimension, see ref. [15].

The fractal dimension of a set is given by

$$D_0=\lim_{\epsilon\to 0}\frac{\log N(\epsilon)}{\log(1/\epsilon)}\,, \tag{8}$$

where, if the set in question is a bounded subset of an m-dimensional Euclidean space R^m, then $N(\epsilon)$ is the minimum number of m-dimensional cubes of side ϵ needed to cover the set. From fig. 2 it can be easily seen that some regions of the attractor are more probable than others. So, in order to understand the dynamics on a chaotic attractor, one must also take into account the distribution or density of points on the attractor. This is more precisely discussed in terms of D_1 which is given by

$$D_1=\lim_{\epsilon\to 0}\frac{I(\epsilon)}{\log(1/\epsilon)}\,, \tag{9}$$

where

$$I(\epsilon)=\sum_{i=1}^{N(\epsilon)} P_i\log(1/P_i) \tag{10}$$

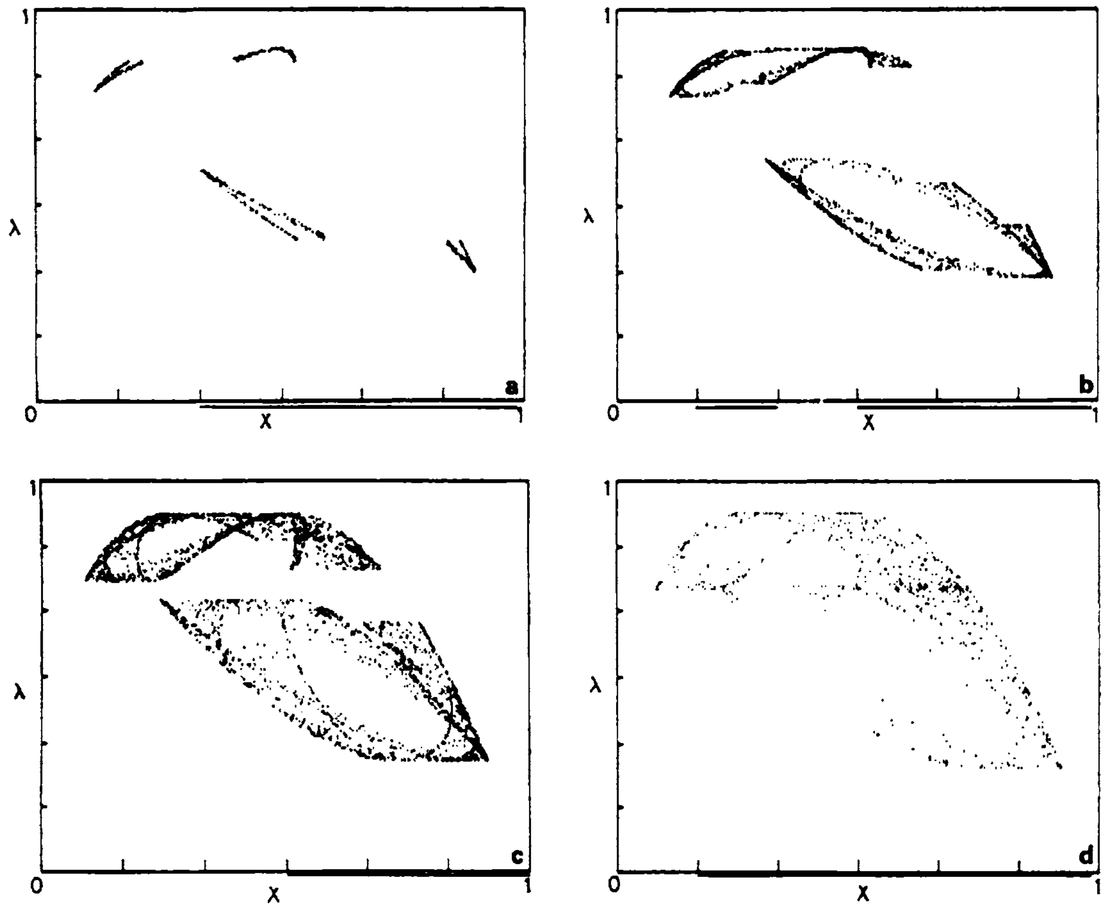


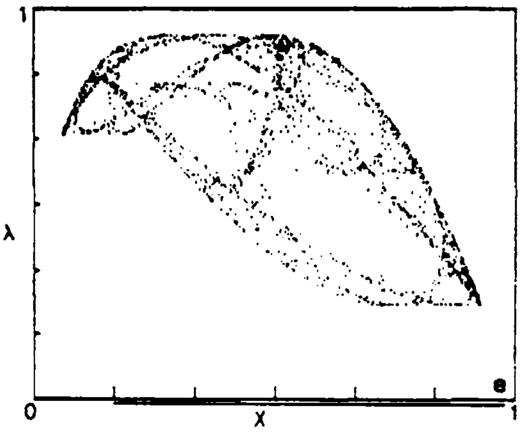
Fig. 2. Strange attractors of our map for parameter values (a) 0.898; (b) 0.903; (c) 0.915; (d) 0.92; (e) 0.93 and (f) 0.98. In each case, 8000 points were used to plot the attractor after discarding the initial 5000 points. Note that initially, the attractor consists of several disjoint sets and merge into a single piece as μ increases.

and P_i is the probability contained within the ith cube. It can be easily shown that $D_0 \geqslant D_1$, where the equality sign holds if the attractor is uniform.

In order to compute these quantities, we made use of the familiar box counting algorithm [15]. The probability P_i is given by η_i/η where η_i is the number of points in each box and η is the total number of points on the attractor. Calculating $N(\epsilon)$ and $I(\epsilon)$ for various values of ϵ and plotting $\ln N(\epsilon)$ and $I(\epsilon)$ separately against $\ln(1/\epsilon)$, D_0 and D_1 were obtained as the asymptotic slopes respectively. Our results are presented in table 1 and in all cases $D_0 > D_1$ as is required.

4. Lyapunov exponents

We know that the dimension of a set depends on its structure or distribution of points in it. For example, the dimension of a Cantor set depends very much on its construction [8]. Strange attractors can exhibit a wide variety of shapes and the complexity of these shapes will be related in some way to the relative amounts of stretching and compression which in turn depends on the LEs. In order to calculate the two LEs of the map (1), let us first consider the Jacobian matrix of the map:



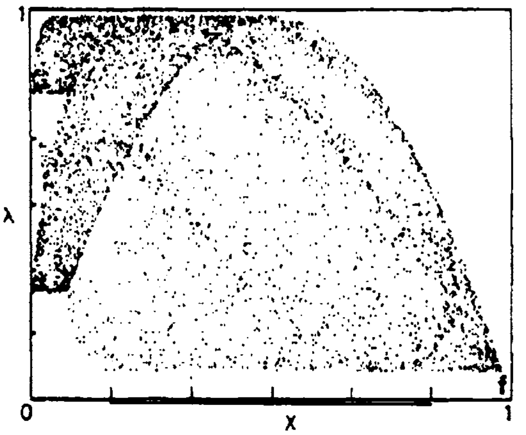


Fig. 2. (continued).

Table 1
Fractal and information dimensions of the strange attractors of the map (1) for various parameter values.

μ	D_0	D_1
0.895	0.94545	0.92920
0.898	1.04571	1.02326
0.9	1.09677	1.08572
0.903	1.20536	1.1500
0.915	1.4400	1.4080
0.92	1.06666	1.05620
0.93	1.23188	1.19492
0.94	1.30435	1.2500
0.98	1.33928	1.32121
0.995	1.58819	1.5160

$$J(X,\lambda)=\begin{pmatrix}4\lambda(1-2X) & 4X(1-X)\\ 0 & 4\mu(1-2\lambda)\end{pmatrix}.$$

The rate of change of an infinitesimal area by the application of the map is given by the determinant of $J(X, \lambda)$ which, in our case, is different at different points along an orbit. Note that the amount by which the area is stretched or compressed along the two coordinate directions are given by the eigenvalues Λ_1 and Λ_2 of $J(X, \lambda)$ since $|J| = \Lambda_1\Lambda_2$. Taking the product of the Jacobian matrices $J(X_i, \lambda_i)$ at N iterates of the map and letting $N\to\infty$, the average rate of stretching or compression along the X and λ directions are given by

$$L_1=\lim_{N\to\infty}\left|\prod_{i=1}^{N}\Lambda_1^{(i)}\right|^{1/N} \tag{11}$$

and

$$L_2=\lim_{N\to\infty}\left|\prod_{i=1}^{N}\Lambda_2^{(i)}\right|^{1/N}, \tag{12}$$

where $\Lambda_1^{(i)}$ and $\Lambda_2^{(i)}$ are simply the diagonal elements of $J(X_i, \lambda_i)$ given vy

$$\Lambda_1^{(i)}=4\lambda_i(1-2X_i) \tag{13}$$

and

$$\Lambda_2^{(i)}=4\mu(1-2\lambda_i)\,. \tag{14}$$

As we know, L_1 and L_2 are called the Lyapunov numbers whose logarithm give the two LEs σ_1 and σ_2. It is easy to see that the LEs of our map are independent of the initial conditions since almost all initial conditions are attracted towards a unique attractor in the unit square. We calculated σ_1 and σ_2 numerically using the above equations for several values of μ and our results are shown in fig. 3. Details of the calculation have already been presented elsewhere [7]. In the figure, we have shown only a few positive values of σ_2 and the corresponding values of σ_1. In between, there are an infinite number of μ values with $\sigma_2<0$ of which only one corresponding to the period-3 window is given. From the variation of σ_1 and σ_2, it becomes clear why the strange attractor becomes more and more stretched out as μ increases.

Now, it is well known that a direct estimation of

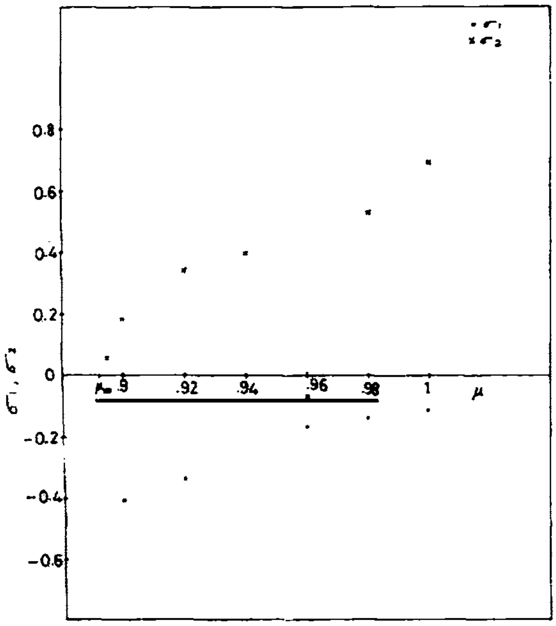


Fig. 3. Lyapunov exponents σ_1 and σ_2 of map (1) for various values of μ. This is only to show the variation of the positive values of σ_2 and the corresponding values of σ_1 which explains why the attractor becomes more and more spread out for increasing values of μ.

the dimension D_1 of the strange attractor can be made in terms of the LEs making use of the Kaplan–Yorke conjecture [16,15]. However, it is found that this is not possible for our map. This may be due to a typical property of noninvertible maps bounded on an interval. Note from fig. 3 that for larger values of μ, $\sigma_1+\sigma_2>0$ which implies that the stretching is globally predominant. For an N-dimensional flow or the related $(N-1)$-dimensional invertible map to be chaotic, the largest LE should be >0 while $\sum_i\sigma_i<0$ [10,17], implying that the phase space volume must contract globally. The main point is that for such systems there is a well defined condition in terms of the LEs which determine the existence of a strange attractor. Now, for noninvertible maps on an interval, this link between the volume contraction and the exponential divergence of nearby trajectories is broken [10]. For such systems, a bounded chaotic motion can occur even if $\sum_i\sigma_i>0$ as is evident from our map. The reason for this depends crucially on the fundamental property of transformation, namely, the noninvertibility. While the sensitive dependence on initial conditions stretches any small initial displacement by an average stretching factor resulting in an exponential increase in the displacement, the noninvertibility helps the mapping to remain bounded in the interval. The process of confinement is exactly analogous to that in the quadratic map which is discussed in detail by Berge et al. [18]. In fact, they show that noninvertibility is essential for a mapping of R into R to be capable of engendering chaos. It then turns out that this fundamental property does play a role along with the sensitive dependence on initial conditions in the formation of strange attractors in our map, making the existence of a direct relationship connecting the fractal dimension and the LEs, such as the one conjectured by Kaplan and Yorke, impossible. It is also worth noting that the conjecture has been shown to hold only for those maps for which the phase space volume contracts after each iteration ($\sum_i\sigma_i<0$) and the contraction is, in fact, uniform everywhere [19,10].

5. Discussion

It is well known that first order nonlinear difference equations such as the logistic map arise naturally in several areas ranging from mathematical economics to population biology [20]. There are of course numerous instances where the dynamics of a complex system is intrinsically multidimensional, in the sense that more than one dynamical variable is needed for a complete specification of the state of the system. This is why the studies on various kinds of coupled and modulated maps [21–24] have been gaining more and more interest recently. We have analysed two coupled first order difference equations of the logistic type which are confined to the unit square. The map shows many interesting properties which are typical of low dimensional chaotic systems. The existence of strange attractors having dimension > 1 indicates that the model may be useful in studying certain currently interesting chaotic phenomena.

It is worth mentioning that the importance of time evolutions with "adiabatically fluctuating parameters (AFPs)" has recently been stressed by Ruelle

[25]. He suggests that the evolution of the parameter may itself be determined by a dynamical system, as in our model. But instead of assuming a slow variation for λ compared to X, we choose the same time scale for X and λ. It is also interesting to note that the same model with $\mu \sim 1$ has been discussed before by Tomita [26], even though in a different context. He used the system as an example of unilateral chaotic modulation to show that the degree of chaos can be reduced by an appropriate modulation or coupling.

Before concluding, we wish to make a special comment regarding the existence of Feigenbaum's universal properties in our map. This result implies that an important factor for the realisation of universal scaling properties is the confinement of the dynamics to an interval, be it in $\mathbb{R}$ or in $\mathbb{R}^2$. As we know, the infinite sequences of period-doubling bifurcations with Feigenbaum scaling have been experimentally observed in several higher dimensional chaotic systems [27,28]. A standard example is the Lorenz model [9] where it is assumed that the solutions of the system, which are identified as the trajectories in phase space, are uniformly bounded as $t \to \infty$. That is, there is a bounded region such that every trajectory ultimately remains with it, analogous to the dynamics in our model. This shows that our result may well be a key in understanding why the universal scaling properties exist in such bounded or "closed flow" systems.

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